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Neutral eigensolutions of the stability equation for stratified shear flow

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Many of the known analytic solutions of the equation for neutral disturbances to a stably stratified, inviscid, parallel shear flow are shown to belong to a wider family of solutions when a transformation to the hypergeometric differential equation is possible. Two particular cases in which the transformation can be made are examined in some detail and the solutions are expressed in a simple analytical form. A number of novel solutions are presented as examples.

1. Introduction

The stability equation for infinitesimal disturbances to the flow of an inviscid, incompressible, stably stratified fluid of density $\rho(z)$ moving with velocity $(U(z), 0, 0)$, when the Boussinesq approximation is made, is

$$\frac{d^2\phi}{dz^2}+\left\{\frac{N^2}{(U-c)^2}-\alpha^2-\frac{U''}{U-c}\right\}\phi = 0, \tag{1}$$

(see, for example, Drazin & Howard 1966, equation 3.12), where z is measured vertically upwards, the stream function $\psi = \phi(z)\exp\{i\alpha(x-ct)\}$, α is the wave-number, $c = c_r+ic_i$ is the (complex) phase speed, and N is the Brunt–Väisälä frequency (so that $N^2 = -g(d\rho/dz)/\rho$). The equation has been non-dimensionalized with respect to an intrinsic length scale L and velocity scale U_0. It is regrettable that this frequently used equation has no name. It seems appropriate to call it the Taylor–Goldstein equation, in recognition of its first use by Taylor (1931) and Goldstein (1931) in the determination of the stability of certain stratified flows, and we shall use this name in this paper. In a number of cases for which an analytic solution can be found, it has been shown that the eigenvalues corresponding to the neutral eigensolutions, the solutions of (1) with $c_i = 0$, subject to certain boundary conditions at $z = z_1$ and z_2, form a stability boundary $J = J_0(\alpha)$, where J_0 is a characteristic value of the Richardson number J $(= N^2/U'^2)$. If a neutral eigensolution can be found, it is frequently possible to test whether or not it lies on a stability boundary by computing $(\partial c/\partial J)_\alpha$ as described by Howard (1963). Recent experiments continuing those described by Thorpe (1968*a*) have shown that the linear stability analysis predicts quite well the features of the onset of instability. It is therefore of interest to find the neutral eigensolutions of (1).

A number of the known analytic neutral solutions of (1) have recently been listed by Drazin & Howard (1966) and it appears that in almost every case the eigensolutions may be immediately deduced if a transformation of (1) into the hypergeometric differential equation is possible. It is therefore worth while to examine the general conditions for which (1) may be transformed into the hypergeometric equation, and this is considered in §2. As a result we shall show that, in certain cases, a continuum of neutral solutions of (1) may be determined by the inverse method of suitably transforming the equation to a hypergeometric differential equation and then determining the forms of $U(z)$ and $\rho(z)$ which lead to solutions of the latter equation. It is found that many of the known analytic solutions belong to families of more general solutions which are described in §3. Further sets of analytic solutions have been discovered for particular cases which may be useful for comparison with experiments or to illustrate or refute conjectures about the stability of stratified flows.

2. The transformation

The Taylor–Goldstein equation, (1), may be written

$$\frac{d^2\phi}{dz^2}+h(z)\phi = 0, \tag{2}$$

where

$$h(z) = \frac{N^2}{(U-c)^2}-\alpha^2-\frac{U''}{U-c}. \tag{3}$$

We shall suppose that the phase speed, c, and therefore $h(z)$, is real, so that the solution corresponds to a neutral eigensolution. We shall specify the functions N^2 and $(U-c)$ in the examples taken later and it has not, in general, been found possible to vary c and U independently. This restriction in the choice of c is a limitation of the method.

The hypergeometric equation is

$$w(1-w)\frac{d^2f}{dw^2}+[m-(k+l+1)w]\frac{df}{dw}-klf = 0, \tag{4}$$

where k, l, m are constants, and Kummer's 24 solutions are listed by Erdélyi (1953) and Abramowitz & Stegun (1965, p. 563). We examine the circumstances in which (2) transforms into (4) together with a transformation of boundary conditions, when

$$\phi = f(w)w^p(1-w)^q, \tag{5}$$

with

$$w = g(z). \tag{6}$$

Substitution of (5) and (6) yields the equation

$$g'^2w^{p-1}(1-w)^{q-1}\left\{w(1-w)\frac{d^2f}{dw^2}+\left[w(1-w)\frac{g''}{g'^2}+2p(1-w)-2qw\right]\frac{df}{dw}\right.$$
$$+\left[p(1-w)\frac{g''}{g'^2}-qw\frac{g''}{g'^2}+p(p-1)\frac{1-w}{w}+q(q-1)\frac{w}{1-w}-2pq\right.$$
$$\left.\left.+w\frac{1-w}{g'^2}h(z)\right]f\right\} = 0, \tag{7}$$

where $$g' \equiv \frac{dg}{dz}, \text{ etc.}$$

Equation (7) is satisfied if the expression within the curly brackets is zero. The terms in this expression have the same form as corresponding terms on the left-hand side of (4) if

$$w(1-w)\frac{g''}{g'^2}+2p(1-w)-2qw = m-(k+l+1)\,w \tag{8}$$

and $$p(1-w)\frac{g''}{g'^2}-qw\frac{g''}{g'^2}+p(p-1)\,\frac{1-w}{w}+q(q-1)\,\frac{w}{1-w}-2pq + \frac{w(1-w)}{g'^2}h = -kl. \tag{9}$$

Equation (8), with $w = g(z)$, is an equation for the function g, and (9), with this function g substituted, is then an equation which must be satisfied by $h(z)$. If $h(z)$ has this required form then the solutions of (2) are related to the solutions of the hypergeometric differential equation (4) by (5). The choices made for p and q are determined by the eigenvalues of (5) since they are related to the constants k, l, and m.

Equation (8) may be written

$$\frac{g(1-g)g''}{g'^2} = Ag+B, \tag{10}$$

where $$A = 2(p+q)-(k+l+1), \quad B = m-2p, \tag{11}$$

and this may be integrated once to give

$$g' = Cg^B/(1-g)^{(A+B)}, \tag{12}$$

where C is a constant. A further integration gives

$$\int \frac{(1-g)^{A+B}}{g^B}\,dg = Cz+(\text{constant}).$$

Substitution from (8) and (12) into (9) gives the equation for $h(z)$,

$$h(z) = \frac{C^2g^{2(B-1)}}{(1-g)^{2(1+A+B)}}\{(2pq-kl)\,g(1-g)+p(1-p)\,(1-g)^2 + q(1-q)g^2+(Ag+B)\,[qg-p(1-g)]\}. \tag{13}$$

There seems little advantage in carrying the general analysis further, and we shall now continue by choosing particular solutions of (10).

3. Particular forms of solution

3.1. *Hyperbolic functions*

Equation (10) is satisfied when $g = \cosh^2 z$, if $A = -1$, $B = \frac{1}{2}$, and (12) is satisfied if, in addition, $C = 2i$, and h then has the form

$$h(z) = \frac{P_1}{\cosh^2 z}+\frac{Q_1}{\sinh^2 z}+R_1, \tag{14}$$

where $P_1 = 2p(2p-1)$, $Q_1 = -2q(2q-1)$, and $R_1 = 4[kl-(p+q)^2]$. (15)

(Equation (14) is the functional form of $h(z)$ which is assumed by (1) when $U = 0$ and $N^2 \propto \operatorname{sech}^2 z$, that is for progressive gravity waves in a fluid with density $\rho = \rho_0 \exp(-\beta \tanh z)$ and this was studied by Groen (1948). Groen used the transformation into the hypergeometric equation in finding solutions; see also Thorpe (1968*b*).)

The constants P_1, Q_1 and R_1, which appear in (14), are related to the physical parameters of the problem by (3). Indeed, when the right-hand side of (3) is equated to that of (14) it is seen that there are related sets of N^2, α and $U-c$, for the particular values of P_1, Q_1 and R_1 for which a solution is possible. If $U-c$ is specified, then the possible forms of N^2 which satisfy the equation are specified. We must now consider the restraints on P_1, Q_1 and R_1 imposed by the possibility of finding a solution.

From (15)
$$p = \tfrac{1}{4}[1 \pm \surd(1+4P_1)], \quad = p^{\pm} \quad \text{say}, \tag{16}$$
$$q = \tfrac{1}{4}[1 \pm \surd(1-4Q_1)], \quad = q^{\pm} \quad \text{say}, \tag{17}$$
and
$$kl = \tfrac{1}{4}[R_1 + 4(p+q)^2], \tag{18}$$
whilst from (11)
$$m = \tfrac{1}{2} + 2p, \tag{19}$$
and
$$k + l = 2(p+q). \tag{20}$$

From this set of equations it may be shown that k or l is equal to
$$p+q \pm \tfrac{1}{2}\surd(-R_1),$$
and without loss of generality we suppose that
$$k = p+q+\tfrac{1}{2}\surd(-R_1) \quad \text{and} \quad l = p+q-\tfrac{1}{2}\surd(-R_1).$$

We shall here pose the problem for an infinite fluid and require that ϕ tends to zero as z tends to $\pm\infty$, that is as w tends to infinity. The general solution for f in the neighbourhood of the singular point at infinity is
$$f(w) = D_0 w^{-k} F(k, k-m+1; k-l+1; 1/w) + D_1 w^{-l} F(l, l-m+1; l-k+1; 1/w), \tag{21}$$
where D_0 and D_1 are arbitrary constants and F is the hypergeometric series.

The solution for ϕ is now found by substituting from (5):
$$\phi = D_2 w^{-\frac{1}{2}\surd(-R_1)}(1-[1/w])^q F(k, k-m+1; k-l+1; 1/w) + D_3 w^{\frac{1}{2}\surd(-R_1)}(1-[1/w])^q F(l, l-m+1; l-k+1; 1/w), \tag{22}$$
where D_2 and D_3 are constants. Since $F(k_1, l_1; m_1; z) \to 1$ as $z \to 0+$ for all k_1, l_1, m_1, it follows that ϕ is bounded as $w \to \infty$ only if $D_3 = 0$, and ϕ then tends to zero at infinity as required. Hence the solution is
$$\phi = D_2 w^{-\frac{1}{2}\surd(-R_1)}(1-[1/w])^q F(k, k-m+1; k-l+1; 1/w). \tag{23}$$

Now (23) represents the solution over only half the required range, either for $z > 0$ or $z < 0$, and we must ensure continuity in ϕ and $d\phi/dz$ across $z = 0$. This is done by demanding that either $\phi = 0$ at $z = 0$ or that $d\phi/dz = 0$ and ϕ bounded and non-zero at $z = 0$. In the first case ($\phi = 0$) continuity is ensured by taking values of D_2 of equal magnitude but opposite sign on opposite sides of $z = 0$, and in the second case ($d\phi/dz = 0$) continuity is ensured by taking the same values of D_2 on either side of $z = 0$.

We shall use the following limits in establishing the conditions in which continuity is ensured:

If $\chi = (1-x)^{\frac{1}{2}n} F(k_1, l_1; m_1; x)$ where $2n = 2(k_1+l_1-m_1)+1$, then, as $x \to 1-$,

$$\chi \to \infty \quad \text{if} \quad \mathscr{R}(n) < 0 \quad \text{or} \quad \mathscr{R}(n) > 1,$$

$$\chi \to \frac{\Gamma(m_1)\,\Gamma(\frac{1}{2})}{\Gamma(m_1-k_1)\,\Gamma(m_1-l_1)} \quad \text{if} \quad \mathscr{R}(n) = 0,$$

$$\chi \to 0 \quad \text{if} \quad 0 < \mathscr{R}(n) < 1,$$

and

$$\chi \to \frac{\Gamma(m_1)\Gamma(\frac{1}{2})}{\Gamma(k_1)\,\Gamma(l_1)} \quad \text{if} \quad \mathscr{R}(n) = 1. \tag{24}$$

These results may easily be deduced by the methods used by Whittaker & Watson (1952; §14.11, and p. 297, example 8). If $\mathscr{R}(l) = 0$ or 1, χ will tend to zero only if the one of the gamma functions appearing in the denominations of the expressions for χ becomes infinite, that is when the argument of the gamma function is equal to a negative integer or zero.

Case 1. $\phi = 0$ *at* $z = 0$. If we write $k_1 = k$, $l_1 = k-m+1$, $m_1 = k-l+1$, and use the relations (19), (20) we find

$$2(k_1+l_1-m_1)+1 = 2(k+l-m)+1 = 4q. \tag{25}$$

Hence, using (24), $\phi \to 0$ as $z \to 0$ (that is, as $1/w \to 1-$) if

$$0 < q < \tfrac{1}{2}, \tag{26}$$

and ϕ is bounded if $q = 0$ or $q = \frac{1}{2}$. For the bounded solutions $\phi \to 0$ as $z \to 0$ if

$$q = 0, \quad \text{and} \quad 1-l = -M \quad \text{or} \quad m-l = -M, \tag{27}$$

or if

$$q = \tfrac{1}{2}, \quad \text{and} \quad k = -M \quad \text{or} \quad k-m+1 = -M, \tag{28}$$

where M is a positive integer or zero. The conditions $q = 0$ and $q = \frac{1}{2}$ are both equivalent to $Q_1 = 0$ as can be seen from (17).

Equations (27) and (28) are equivalent to the condition

$$k = p+q+\tfrac{1}{2}\sqrt{(-R_1)} = -M, \tag{29}$$

provided that both the positive and negative values in the expressions (16) and (17) for p and q are considered.

The condition (26) is satisfied when

$$0 < Q_1 \leqslant \tfrac{1}{4}. \tag{30}$$

Case 2. $d\phi/dz = 0$ *and* ϕ *is bounded and non-zero at* $z = 0$. In case 1 we showed that ϕ can only be bounded and non-zero if $q = 0$ or $\frac{1}{2}$ and so only these cases need be considered here.

Now

$$w^{-k}F(k, k-m+1; k-l+1; 1/w) = w^{l-m}(w-1)^{m-k-l}F(1-l, m-l; k-l+1; 1/w)$$

(Abramowitz & Stegun 1965, equation 15.5.7), and so ϕ may be written

$$\phi = D_2 w^{p+l-m} F(1-l, m-l; k-l+1; 1/w),$$

and
$$\frac{d\phi}{dw} = \left(\frac{1}{w}\right)(p+l-m)\phi + D_2 w^{p+l-m}\frac{dF}{dw}. \tag{31}$$

Now $d\phi/dz$ is found by multiplying $d\phi/dw$ by $dw/dz = 2w^{\frac{1}{2}}(w-1)^{\frac{1}{2}}$; since ϕ is bounded the product of the first term on the right-hand side of (31) with dw/dz tends to zero as $w \to 1-$ (that is as $z \to 0$). Hence $d\phi/dz$ tends to zero as $z \to 0$ if $w^{p+l-m+\frac{1}{2}}(w-1)^{\frac{1}{2}}\, dF/dw \to 0$ as $w \to 1-$. Now

$$w^{p+l-m+\frac{1}{2}}(w-1)^{\frac{1}{2}}(d/dw)\,F(1-l, m-l; k-l+1; 1/w)$$
$$= -(w-1)^{\frac{1}{2}} w^{p+l-m-\frac{3}{2}} \frac{(1-l)(m-l)}{k-l+1} F(2-l, 1+m-l; k-l+2; 1/w) \tag{32}$$

(see Abramowitz & Stegun 1965, equation 15.2.1; $k-l+1 = 1+\sqrt{(-R_1)} \neq 0$) and the right-hand side of (32) tends to

$$-\frac{(1-l)(m-l)\,\Gamma(k-l+2)\,\Gamma(\frac{1}{2})}{(k-l+1)\,\Gamma(2-l)\,\Gamma(m-l+1)},$$

as $w \to 1-$, as is easily shown by using (24). Hence $d\phi/dz \to 0$ as $z \to \infty$, if $l = 1$, $m = l$, $2-l = -M$ or $m-l+1 = -M$, and by substitution of the expressions (19) and (20) it is easily seen that these conditions are included in the set of conditions (29) when p and q take all possible values.

Hence, in general, we have the continuum solution (see also Case 1960), continuous at $z = 0$, and tending to zero as $z \to \pm\infty$,

$$\phi = D_2(\operatorname{sech} z)^{\sqrt{(-R_1)}}(\tanh z)^{2q} F(p+q+\tfrac{1}{2}\sqrt{(-R_1)}, [\tfrac{1}{2}-p]+q+\tfrac{1}{2}\sqrt{(-R_1)}; \operatorname{sech}^2 z), \tag{33}$$

with $0 < q < \frac{1}{2}$ (and therefore $0 < Q_1 \leqslant \frac{1}{4}$), or the solutions with $q = 0$ or $\frac{1}{2}$ (and therefore $Q_1 = 0$) and $p+q+\frac{1}{2}\sqrt{(-R_1)} = -M$, where M is a positive integer or zero. The latter solution is that which corresponds to the degenerate form of the solution of the hypergeometric equation, when the series F terminates after $M+1$ terms. The solutions ϕ may in general be expressed in simple analytical terms when F terminates and, for illustration, we shall take examples corresponding to the degenerate form of the solution and impose the condition (29). These solutions will form a family belonging to the continuum solution and, for terms of reference, we shall call them the degenerate solutions. In some particular examples it will be found that the degenerate solutions form all, or part of, a stability boundary. The other solutions will be of use in the description of general disturbances to the flow. It will usually be possible to satisfy (29) for only a few values of M.

3.2. *Examples: h(z) given by* (14)

Some examples are listed below. These are constructed by choosing expressions for N^2 (or for $U-c$) and finding the corresponding forms of $(U-c)$ (or N^2), when $h(z)$ is given by (3) and (14). The constant D_2 has been set equal to unity.

Example 1. $N^2 = J_0 \operatorname{sech}^{2n} z$, $U-c = \operatorname{sech}^n z \tanh z$; $n \geqslant 0$, $J_0 \geqslant 0$. Here

$$h(z) = \frac{(n+1)(n+2)}{\cosh^2 z} + \frac{J_0}{\sinh^2 z} + J_0 - \alpha^2 - n^2,$$

$$p^{\pm} = \tfrac{1}{4}[1 \pm (2n+3)] \quad \text{and} \quad q^{\pm} = \tfrac{1}{4}[1 \pm \surd(1-4J_0)].$$

Solutions of (29) are possible only if $M = 0$, $p = -\frac{1}{2}n(n+1)$,

$$q = \tfrac{1}{2} - \frac{\alpha^2}{2(1+2n)} \quad \text{and} \quad J_0 = \frac{\alpha^2}{(1+2n)^2}(1+2n-\alpha^2)$$

(the eigenvalue equation), with $0 \leqslant \alpha^2 \leqslant 1+2n$. The neutral eigensolution (the degenerate solution) is $\phi = \cosh^{-(1+n)} z \,|\sinh z\,|^{[1-\alpha^2/(1+2n)]}$.

The continuum solution is possible if $0 < J_0 \leqslant \frac{1}{4}$.

When $n = 0$ the solution found by Drazin (1958) is recovered, whilst when $n = 1$ we have the solution found by Drazin & Howard (1966, p. 76).

Example 2. $N^2 = J_0 \operatorname{sech}^{2(1+n)} z$, $U-c = \operatorname{sech}^n z \tanh z$; $n \geqslant -\frac{1}{2}$, $J_0 \geqslant 0$. Here

$$h(z) = \frac{(n+1)(n+2)}{\cosh^2 z} + \frac{J_0}{\sinh^2 z} - \alpha^2 - n^2,$$

and $p^{\pm}$, $q^{\pm}$ are given as in example 1.

Solutions of (29) are possible only if $M = 0$, $p = -\frac{1}{2}(n+1)$ and

$$q = \tfrac{1}{2}[1+n-\surd(\alpha^2+n^2)],$$

and
$$J_0 = [\surd(\alpha^2+n^2)-n][1+n-\surd(\alpha^2+n^2)]$$

(the eigenvalue equation), with $0 \leqslant \alpha \leqslant 1$. The neutral eigensolution (the degenerate solution) is

$$\phi = \cosh^{-(1+n)} z \,|\sinh z\,|^{\frac{1}{2}[1+n-\surd(\alpha^2+n^2)]}.$$

The continuum solution is possible if $0 < J_0 \leqslant \frac{1}{4}$. When $n = 0$ we have Holmboe's (1962) solution.

Example 3. $N^2 = J_0 + J_1 \tanh^2 z$, $U-c = \sinh z$; J_0 must be non-negative to satisfy the condition $N^2 \geqslant 0$ at $z = 0$. Here

$$h(z) = \frac{J_1}{\cosh^2 z} + \frac{J_0}{\sinh^2 z} - 1 - \alpha^2$$

and
$$p^{\pm} = \tfrac{1}{2}[1 \pm \surd(1+4J_1)], \quad q^{\pm} = \tfrac{1}{4}[1 \pm \surd(1-4J_0)].$$

Equation (29) may be satisfied if

$$J_0 = \tfrac{1}{4}\{1-[2+4N-\surd(1+4J_1)+2\surd(1+\alpha^2)]^2\},$$

where the modulus of the terms in the square bracket must be less than or equal to unity. It follows that, for any degenerate solution to be possible,

$$J_1 \geqslant 2(2M+1)(M+1) \quad (M \geqslant 0).$$

For the continuum solution, $0 < J_0 \leqslant \frac{1}{4}$.

For example, if $J_1 = 6$, M must be zero, and the degenerate solution is

$$J_0 = [\sqrt{(1+\alpha^2)}-1][2-\sqrt{(1+\alpha^2)}] \quad (\sqrt{3} \geqslant \alpha \geqslant 0), \tag{34}$$

and $\phi = \operatorname{sech}^2 z \,|\sinh z\,|^{[2-\sqrt{(1+\alpha^2)}]}$.

Using the techniques described by Howard (1963) it is found that, for $c_r = 0$,

$$(\partial c/\partial J)_\alpha = \frac{-10i\gamma \cot[\pi(\gamma-1)]\,B(\frac{5}{2}-\gamma,\gamma)}{B(2-\gamma,\gamma+\frac{1}{2})(10\gamma^2+9\gamma-3)(3-2\gamma)}$$

on the neutral curve (34), where $\gamma = \sqrt{(1+\alpha^2)}$ and B is the beta function. The expression $(\partial c/\partial J)_\alpha$ is always imaginary and negative on the neutral curve and so (34) is a stability boundary with instability for $J < J_0$, $c_r = 0$. This result has been confirmed by numerical calculations of Mr Philip Hazel at Cambridge, who has also considered the effects of boundaries at finite z. This is an example of a flow which, if unstratified, is stable by Fjørtoft's (1950) criterion, but which is destabilized by the addition of a stable density stratification. Another well-known example is that with a three-layer density structure and $U = z$, which was found and explained by Taylor (1931). In that case the instability was explained as being due to a resonance between a wave moving backwards relative to the basic flow at the upper interface and a wave moving forward relative to the basic flow at the lower interface, when the absolute speed of the waves was approximately the same. In this case no such simple explanation is possible, although it seems likely that a similar mechanism causes instability.

If now $J_1 = 20$, there are two degenerate solutions.

For $M = 0$, $J_0 = [\sqrt{(1+\alpha^2)}-3][4-\sqrt{(1+\alpha^2)}] \quad (15 \geqslant \alpha^2 \geqslant 8)$,

and
$$\phi = \cosh^{-4} z\,|\sinh z\,|^{4-\sqrt{(1+\alpha^2)}};$$

whilst, for $M = 1$,

$$J_0 = [2-\sqrt{(1+\alpha^2)}][\sqrt{(1+\alpha^2)}-1] \quad (3 \geqslant \alpha^2 \geqslant 0),$$

and
$$\phi = (\operatorname{sech} z)^{\sqrt{(1+\alpha^2)}}\,|\tanh z\,|^{[2-\sqrt{(1+\alpha^2)}]}\left\{1-\frac{7}{2[1+\sqrt{(1+\alpha^2)}]}\operatorname{sech}^2 z\right\}.$$

There are many other solutions which may be constructed as particular cases. These include the profile

$$N^2 = J_0(1-r+3r\tanh^2 z)\operatorname{sech}^2 z, \quad U = \tanh z,$$

examined by Miles (1963), which itself includes as special cases the solutions of Holmboe ($r = 0$) and Garcia ($r = 1$), and some jet-type solutions.†‡

† For example $U-c = \operatorname{sech}^2 z$, $N^2 = J_0 \operatorname{sech}^4 z$; this has the (degenerate) neutral eigensolutions $\phi = \tanh z \operatorname{sech}^2 z$, $J_0 = 3+\alpha^2$, and $\phi = \operatorname{sech}^2 z$, $J_0 = \alpha^2$. These do not appear to be stability boundaries.

‡ Miles (1967) has also used a transformation to hypergeometric form for the solution of a problem in which $N^2 = J_0\, e^{-z}$, $U = 1-e^{-z}$.

3.3. *Trigonometric functions*

Equation (10) is satisfied when $g = \cos^2 z$ if $A = -1, B = \frac{1}{2}$, and (12) if $C = -2$, and h then has the form

$$h(z) = \frac{P_2}{\cos^2 z} + \frac{Q_2}{\sin^2 z} + R_2, \tag{35}$$

where $\quad P_2 = 2p(1-2p), \quad Q_2 = 2q(1-2q) \quad$ and $\quad R_2 = 4[(p+q)^2 - kl]. \quad (36)$

If we solve as before, we find the solution

$$\phi = E_1(\cos z)^{2p}(\sin z)^{2q} F(k,l;m;\cos^2 z)$$
$$+E_2(\cos z)^{1-2p}(\sin z)^{2q}F(k-m+1, l-m+1; 2-m; \cos^2 z),$$

where, to ensure the boundary condition $\phi \to 0$ as $\cos z \to 0$, the constant $E_1 = 0$, if $p \leqslant 0$, and the constant $E_2 = 0$, if $p \geqslant \frac{1}{2}$. The function ϕ tends to zero as $\cos^2 z \to 1-$, if $0 < Q_2 \leqslant \frac{1}{4}$, and is continuous if $Q_2 = 0$ and $p + q \pm \frac{1}{2}\sqrt{R_2} = -M$, where M is a positive integer or zero. We shall define the degenerate solutions to be solutions in which

$$p + q \pm 4\sqrt{R_1} = -M, \tag{37}$$

and take special cases as before.

3.4. *Examples: h(z) given by* (35)

Example 1. $N^2 = J_0$, $U - c = \sin z; J_0 \geqslant 0$. Here $h(z) = (J_0/\sin^2 z) + 1 - \alpha^2$ and so $p^{\pm} = 0$ or $\frac{1}{2}$ and $q^{\pm} = \frac{1}{4}[1 \pm \sqrt{(1-4J_0)}]$.

Equation (37) can only be satisfied for $M = 0$ and with

$$J_0 = \sqrt{(1-\alpha^2)} + \alpha^2 - 1 \quad (\alpha^2 \leqslant 1), \tag{38}$$

and the degenerate eigensolution is

$$\phi = |\sin z|^{\sqrt{(1-\alpha^2)}},$$

and this is a solution found by Drazin & Howard (1966, equation 5.36 i). The second solution (equation 5.36 ii) found by Drazin & Howard is satisfied only by negative J_0, and their third solution (equation 5.36 iii) has the correct eigensolution (this is not given by the present theory as the solution is not expressible in the form we have assumed) but the incorrect eigenvalue. The solution should be

$$\phi = |\cos(z/2)|^{\frac{1}{2} \pm \sqrt{(\frac{1}{4} - J)}} |\sin(z/2)|^{\frac{1}{2} \mp \sqrt{(\frac{1}{4} - J)}} \quad \text{with} \quad \alpha = \tfrac{1}{2}\sqrt{3} \quad (0 \leqslant J_0 < \tfrac{1}{4}).\dagger$$

Mr Hazel has examined the stability of the flow in this case with boundaries at $z = -\pi, \pi$, using a numerical technique, and has found the flow to be unstable for $0 \leqslant J_0 < \sqrt{(1-\alpha^2)} + \alpha^2 - 1$, $0 < \alpha < \frac{1}{2}\sqrt{3}$, and stable elsewhere. The neutral curve (29) is thus a stability boundary over only that part of its path for which $0 < \alpha < \frac{1}{2}\sqrt{3}$. The instability for $0 < \alpha < \frac{1}{2}\sqrt{3}$ with $J_0 = 0$ was discovered by Tollmien (1935).

† If $h(z) = (Q_3/\sin^2 z) + R_3$ it may easily be shown by substitution that a solution of (2) is $R_3 = \frac{1}{4}$, $Q_3 \leqslant \frac{1}{4}$, and $\phi = |\cos(z/2)|^{\frac{1}{2} \pm \sqrt{(\frac{1}{4} - Q_3)}} |\sin(z/2)|^{\frac{1}{2} \mp \sqrt{(\frac{1}{4} - Q_3)}}$.

Example 2. $N^2 = J_0 \cos^2 z$, $U-c = \sin z$; $J_0 \geqslant 0$. Here

$$h(z) = J_0/\sin^2 z + 1 - \alpha^2 - J_0$$

and so $p^{\pm} = 0$ or $\frac{1}{2}$ and $q^{\pm} = \frac{1}{4}[1 \pm \sqrt{(1-4J_0)}]$. Equation (37) can only be satisfied for $M = 0$ and with $J_0 = \alpha^2(1-\alpha^2)$, $\alpha^2 \leqslant 1$, and the degenerate eigensolution is $\phi = |\sin z|^{(1-\alpha^2)}$. (As in example 1 a second neutral eigensolution exists; see footnote.)

Example 3. $N^2 = J_0 \sin^2 z$, $U-c = \sin z$; $J_0 \geqslant 0$. Here $h(z) = J - \alpha^2 + 1$ and the solution with boundaries at $z = z_1, z_2$ is

$$\phi = \sin\left[n\pi\left(\frac{z-z_1}{z_2-z_1}\right)\right],$$

where

$$J_0 = \alpha^2 + \frac{n^2\pi^2}{(z_2-z_1)^2} - 1$$

and n is an integer.

In this case $Q_2 = 0$ and the continuum solution does not exist.

4. Final remarks

We have shown that when the expression

$$h(z) = \left\{\frac{N^2}{(U-c)^2} - \alpha^2 - \frac{U''}{U-c}\right\},$$

which appears in the Taylor–Goldstein equation, (1), takes certain forms (which are given in general by (13), where g is defined by (10)), the solutions ϕ belong to continua which may be expressed in analytical forms. Two forms of $h(z)$, (14) and (35), have been considered in some detail and provide examples of neutral eigensolutions, many of which have not previously been noticed.

The possible solutions of (10) have not been exhausted by the two particular forms of solution chosen above, but these serve to illustrate the method of finding solutions of (1).

If the minimum Richardson number is greater than $\frac{1}{4}$, the flow is known to be stable (Miles 1961). No cases have been found here in which the criteria for instability is that the minimum Richardson number in the flow is less than a positive number *less than* $\frac{1}{4}$, but an example has been found which shows that stability of a stratified flow is not ensured by the stability of an unstratified flow with the same velocity distribution.

Although transformation into the hypergeometric differential equation serves to provide a good example, any transformation of the Taylor–Goldstein equation into a form of which solutions are known will, of course, generate a further class of solutions.

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